The Black-Scholes-Merton Approach to Pricing Options

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Introduction

In these notes we shall now discuss the Black-Scholes-Merton approach to determining the fair price of an option using the principles of no arbitrage. The key idea will be to show that for an option with a given payoff a dynamic trading strategy can be devised, when starting with some initial portfolio, which replicates the option's payoff. The value of the initial portfolio will then determine the price of the option by the principle of no arbitrage. We will carry out this analysis in the context of a basic model for the stock price dynamics which requires only a single parameter be estimated from the market place, namely the volatility. We will then derive the well-known Black-Scholes-Merton Formula for the European call and put options. From this formulation the Black-Scholes-Merton PDE is then derived for the case of a European option.

A Multiperiod Stock Price Model

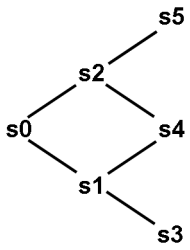

Figure 1: Multiperiod Model for the Stock

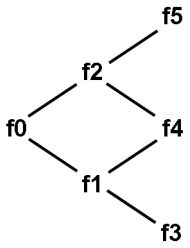

Figure 2: Multiperiod Model for the Option

We shall now consider a multi-period model for the stock price dynamics over time, see Figure 1 for an example of a two period model. For the market we shall assume there is always the opportunity to invest in a risk-free asset (bond) which has a compounding return (interest rate) r. The principle of no arbitrage must hold for each individual period in the model, which requires that $s_{\text{down}} < s_{\text{now}} e^{r\Delta t} < s_{\text{up}}$. Let us denote for a model with N periods, the time elapsed each period by $\Delta t = T/N$, then the jth period corresponds to the time period $[t_j, t_{j+1}]$, where $t_j = j\Delta t$.

Let us now consider a contingent claim in the market which depends only on the stock price at the maturity time T. The value of the claim then depends on the current state of the market place, as in Figure 2. Since the payoff of the contingent claim at maturity time T is known, in the case of a call $(s-K)_+$ and in the case of a put $(K-s)_+$, the problem remains to determine the value of the contingent claim at times before maturity and in particular today's price f_0.

A natural approach is to use the same ideas as we discussed for the single period binomial market model. If we consider a particular period, a portfolio consisting of a bond and stock can be constructed which replicates the value of the option over the period. This requires that we find weights w_1 for the stock and

w_2 for the bond to determine a portfolio which at the end of the period has a value satisfying:

$$\begin{aligned} w_1 s_{\text{up}} + w_2 &= f_{\text{up}} \\ w_1 s_{\text{down}} + w_2 &= f_{\text{down}}. \end{aligned}$$

This has the solution

$$\begin{aligned} w_1^* &= \frac{f_{\text{up}} - f_{\text{down}}}{s_{\text{up}} - s_{\text{down}}} \\ w_2^* &= \frac{s_{\text{up}} f_{\text{down}} - s_{\text{down}} f_{\text{up}}}{s_{\text{up}} - s_{\text{down}}}. \end{aligned}$$

The value of the portfolio at the beginning of the period is then known and must agree with the value of the contingent claim by the principle of no-arbitrage, which gives the value:

$$f_{\text{now}} = w_1^* s_{\text{now}} + w_2^* e^{-r\Delta t}.$$

By rearranging the terms this can be expressed as:

$$f_{\text{now}} = e^{r\Delta t} \left[q f_{\text{up}} + (1-q) f_{\text{down}} \right]$$

where

$$q = \frac{e^{r\Delta t} s_{\text{now}} - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}.$$

This now allows for us to obtain the value of the contingent claim for all stock prices right before the maturity. By working inductively using this formula the value of the contingent claim at all values of the tree can be determined. If we assume that the stock prices are such that q is the same over a each period we obtain for f_0 at time 0 in the two period model the price (see Figure 2):

$$f_0 = e^{-2r\Delta t} \left[(1-q)^2 f_3 + 2q(1-q) f_4 + q^2 f_5 \right].$$

An important feature of the expression we derived inductively is that the value at f_0 only depends on the payoff values of the contingent claim at maturity, see Figure 2.

To ensure that q is the same for each period we shall now assume that the multi-period market model has the structure that $s_{\text{up}} = u s_{\text{now}}$ and $s_{\text{down}} = d s_{\text{down}}$, where u and d are the same for each period. Although more general models can be considered for our discussion the formulas become more complicated. The value of the contingent claim for this type of multi-period binomial market with N periods is given by:

$$f_0 = e^{-N\Delta t} \left[\sum_{k=0}^{N} \binom{N}{k} q^k (1-q)^{N-k} f(s_T(k)) \right] \tag{1}$$

where the payoff of the contingent claim is given by $f(s)$ and the value of the stock at maturity by:

$$s_T(k) = u^k d^{N-k} s_0.$$

For this market model we have:

$$q = \frac{e^{r\Delta t} - d}{u - d}$$

and the condition for no arbitrage becomes $d < e^{r\Delta t} < u$.

The value of the option can then be expressed as:

$$f_0 = e^{-rT} \mathrm{E} \left(f(S_T) \right)$$

where the S_T is a random variable having the binomial distribution. In other words, for this market model the principles of no arbitrage give that the "risk-neutral" probability for pricing contingent claims is the stock price the binomial distribution.

Lognormal Stock Price Dynamics

We shall now consider a continuous time limit $\Delta t \to 0$ of the model for the stock market prices. To obtain a finite limit we need to specify some type of dependence of the factors $u(\Delta t)$ and $d(\Delta t)$ on the duration of the time period. A natural choice is to consider the effective continuous return ρ of an asset, which can be readily compared to continuously compounding interest rates. This is defined over the period $[0, \Delta t]$ by

$$s(\Delta t) = s(0)e^{\rho \Delta t}.$$

We can solve for ρ to express this as:

$$\rho = \frac{\log(S(\Delta t)) - \log(S(0))}{\Delta t}.$$

The effective continuous return $\rho \Delta t$ of an asset is uncertain at time 0 and the mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$ are typically used to characterize the random return over the period Δt. In other words, the effective continuous return $\rho \Delta t$ over the period will on average be $\mu \Delta t$ while the fluctuations of the return will typically be of a size comparable to $\sigma\sqrt{\Delta t}$. Using these statistics a natural approach to model the fluctuations of the stock price over the period Δt in the binomial setting is:

$$\begin{aligned} \rho_{\text{up}} \Delta t &= \mu \Delta t + \sigma\sqrt{\Delta t} \\ \rho_{\text{down}} \Delta t &= \mu \Delta t - \sigma\sqrt{\Delta t}. \end{aligned}$$

Since $s_{\text{up}} = e^{\rho_{\text{up}} \Delta t} s_{\text{now}}$ and $s_{\text{down}} = e^{\rho_{\text{down}} \Delta t} s_{\text{now}}$ this motivates using the factors:

$$\begin{aligned} u(\Delta t) &= e^{\mu \Delta t + \sigma\sqrt{\Delta t}} \\ d(\Delta t) &= e^{\mu \Delta t - \sigma\sqrt{\Delta t}}. \end{aligned}$$

This gives the valuation of the option over a period:

$$f_{\text{now}} = e^{-r\Delta t} \left[q f_{\text{up}} + (1-q) f_{\text{down}} \right] \tag{2}$$

where the "risk-neutral" probability is:

$$q = \frac{e^{(r-\mu)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

The valuation of the option in terms of the payoff is given in equation 1 and is a binomial distribution. From equation 2 the stock value $s_T(k)$ after N periods when there are k up-ticks of the price can be expressed as:

$$s_T(k) = e^{\mu N \Delta t - (N-2k)\sigma\sqrt{\Delta t}}.$$

Since the stock price at maturity has a binomial distribution where the probability of an up-tick in the stock price is q we can express the stock price random variable S_T as:

$$S_T^{(N)} = \exp\left[\mu T + \sum_{j=1}^{N} X_j \sigma\sqrt{\Delta t} \right] s_0.$$

where the X_j are independent random variables with

$$X_j = \begin{cases} +1, \text{ with probability } q \\ -1, \text{ with probability } 1-q \end{cases}.$$

Now by using that $\Delta t = T/N$ and some algebra we can express the stock price random variable as:

$$S_T^{(N)} = \exp\left[\mu T + \left(\frac{\sum_{j=1}^N X_j - m_0^{(N)}}{\sqrt{N}}\right)\sigma\sqrt{T} + m_0^{(N)}\sigma\sqrt{\Delta t}\right] s_0 \tag{3}$$

where

$$m_0^{(N)} = \mathrm{E}\left(X_j\right) = 2q - 1.$$

Now by Taylor expansion in Δt we can approximate q by:

$$q = \frac{1}{2} + \frac{(r - \mu - \frac{1}{2}\sigma^2)}{2\sigma}\sqrt{\Delta t} + \cdots. \tag{4}$$

This yields the approximation

$$m_0^{(N)} = \frac{(r - \mu - \frac{1}{2}\sigma^2)}{\sigma}\sqrt{\Delta t}.$$

Substituting this into equation 3 gives:

$$S_T^{(N)} = \exp\left[(r - \frac{1}{2}\sigma^2)T + \left(\frac{\sum_{j=1}^N X_j - m_0^{(N)}}{\sqrt{N}}\right)\sigma\sqrt{T}\right] s_0.$$

We remark that the model to leading order no longer depends on the drift specified for the stock price. In particular, no arbitrage priciples in determining the "risk-neutral" probability compensates for the influence of this parameter in the model, making the only parameter required for the stock the market volatility σ.

We are now ready to consider the process in the limit as $\Delta t \to 0$. By the Central Limit Theorem (which can be readily extended to deal with $m_0^{(N)} \to 0$) we have that:

$$\frac{\sum_{j=1}^N X_j - m_0^{(N)}}{\sqrt{N}} \to Z$$

where $Z \sim \eta(0,1)$ denotes a standard Gaussian random variable with mean 0 and variance 1. This gives in the limit for the stock price random variable that:

$$S_T^{(N)} \to S_T = \exp\left[(r - \frac{1}{2}\sigma^2)T + Z\sigma\sqrt{T}\right] s_0.$$

Again,we remark that in the $\Delta t \to 0$ limit the stock price model only depends on σ and not the specified drift μ. However, for non-zero Δt the drift μ does play a role, and by equation 4 the convergence of the discrete model to the continuum limit can be accelerated by choosing $\mu = r - \frac{1}{2}\sigma^2$. We remark that the $\log(S_T) \sim \eta(r - \frac{1}{2}\sigma^2, \sigma^2 T)$ which is a Gaussian with respective mean $(r - \frac{1}{2}\sigma^2)$ and variance $\sigma^2 T$. This type of distribution for the stock price is referred to as lognormal.

Now the "risk-neutral" valuation of the option in the continuum limit becomes:

$$f_0 = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\left(e^{(r-\frac{1}{2}\sigma^2)T + z\sigma\sqrt{T}} s_0\right) e^{-\frac{z^2}{2}} dz. \tag{5}$$

Black-Scholes-Merton Formula

We use the general option pricing formula above, equation 5, to price the call and put options with payoffs $(s-K)_+$ and $(K-s)_+$, respectively. This gives the Black-Scholes-Merton Formula for the call and put:

$$\begin{aligned} c(s_0, K, T) &= s_0 N(d_1) - K e^{-rT} N(d_2) \\ p(s_0, K, T) &= K e^{-rT} N(-d_2) - s_0 N(-d_1) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{s_0}{K}\right) + (r + \frac{1}{2}\sigma^2)T\right) \\ d_2 &= \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{s_0}{K}\right) + (r - \frac{1}{2}\sigma^2)T\right). \end{aligned}$$

We remark that $d_2 = d_1 - \sigma\sqrt{T}$. In the notation we denote the cumulative distribution function of the standard normal distribution by:

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{y^2}{2}}dy.$$

When considering the option price an important issue is the sensitivity of the price on the parameters of the market model. This is reflected in the partial derivatives of the option price, which in finance are referred to as the "Greeks", which have the definitions:

Definition:	Call	Put
Delta $= \frac{\partial}{\partial s_0}$	$N(d_1) > 0$	$-N(-d_1) < 0$
Gamma $= \frac{\partial^2}{\partial s_0^2}$	$\frac{1}{s_0\sigma\sqrt{2\pi T}}\exp\left(-\frac{d_1^2}{2}\right) > 0$	$\frac{1}{s_0\sigma\sqrt{2\pi T}}\exp\left(-\frac{d_1^2}{2}\right) > 0$
Theta $= \frac{\partial}{\partial t}$	$-\frac{s_0\sigma}{2\sqrt{2\pi T}}\exp\left(-\frac{d_1^2}{2}\right) - rKe^{-rT}N(d_2) < 0$	$-\frac{s_0\sigma}{2\sqrt{2\pi T}}\exp\left(-\frac{d_1^2}{2}\right) + rKe^{-rT}N(-d_2)$ (either sign)
Vega $= \frac{\partial}{\partial \sigma}$	$\frac{s_0\sqrt{T}}{\sqrt{2\pi}}\exp(-\frac{d_1^2}{2}) > 0$	$\frac{s_0\sqrt{T}}{\sqrt{2\pi}}\exp(-\frac{d_1^2}{2}) > 0$
Rho $= \frac{\partial}{\partial r}$	$TKe^{-rT}N(d_2) > 0$	$-TKe^{-rT}N(-d_2) < 0$

Consideration of the "Greeks" is especially important when attempting to hedge against changes in the value of a portfolio when changes occur in the market-place. For example, suppose we invest W amount of wealth in a portfolio consisting of w_1 units of a stock s, w_2 units of a bond b, and one short unit of a call option c with the same underlying type of stock. The total value of the portfolio is:

$$V = w_1 s + w_2 b - c.$$

Suppose that we would like to construct this portfolio so that the value changes little when small changes occur in the stock price. This can be accomplished by setting the derivative of V in s to zero:

$$\frac{\partial V}{\partial s} = w_1 - \text{Delta} = 0.$$

Thus, if we invest $w_1 = \text{Delta}$ units in the stock and $w_2 = W - \text{Delta} * s$ in the bond, so the total value of the portfolio will change very little for small changes in the stock price.

We remark that such a portfolio and hedge is useful for example when a bank sells a call option. The proceeds from selling the option must be invested so that the bank can fulfill its obligations at maturity. From no arbitrage principles we showed that there is a self-financing dynamic replicating portfolio strategy involving a stock and bond which matches the value of the option. By the hedging argument above the weights of this replicating portfolio at each stock price can be expressed in terms of the option's Delta. An interesting issue that arises in hedging the value of an option in practice is that the replicating portfolio can only be adjusted at discrete times. The Greeks can be used to estimate the error which occurs as a result of the discrete time hedging. For example, to estimate the error in hedging changes in the stock price the Gamma can be used. For example, if the replicating portfolio V_0 exactly matches the value of the option at time 0, when the stock price is s_0, then the error in replicating the change in the option value over a short period of time, where say $s_1 = s(\Delta t)$, by holding the portfolio, which has value V_1 with the new stock price, can be estimated. By using the Taylor expansion of $c(s)$ we have:

$$V_1 - V_0 = w_1 s_0 + w_2 b - c(s_0) - (w_1 s_1 + w_2 b - c(s_1))$$

$$
\begin{aligned}
&= w_1(s_1 - s_0) - \frac{\partial c}{\partial s}(s_0)(s_1 - s_0) - \frac{1}{2}\frac{\partial^2 c}{\partial s^2}(s_0)(s_1 - s_0)^2 + \cdots \\
&= -\frac{1}{2}\text{Gamma} \cdot (s_1 - s_0)^2 + \cdots
\end{aligned}
$$

where $w_1 = \text{Delta}$ cancels the first derivative in c. This indicates the hedging error that will be accumulated over each holding period when adjusting the replicating portfolio only at the discrete times. We shall discuss more about the implications of this result in future lectures.

Stochastic Calculus

Many continuous stochastic processes X_t arise from an appropriately scaled limit of a discrete random process, similar to that described above. As a consequence of the central limit theorem, many of these stochastic processes can be related to the Wiener process B_t, also referred to as Brownian motion. The stochastic process B_t has the following features:
(i) $B_0 = 0$.

(ii) B_t is a continuous function.

(iii) The increment $B_{t_2} - B_{t_1}$ has a Guassian distribution with mean 0 and variance $t_2 - t_1$.

(iv) The increments $B_{t_2} - B_{t_1}$ and $B_{t_4} - B_{t_3}$ are independent for $t_1 < t_2 < t_3 < t_4$.

For a more rigorous discussion see [1;2].

A stochastic differential equation attempts to relate a general stochastic process to Brownian motion by expressing the increments in time of the stochastic process in terms of the increments of Brownian motion:

$$dX_t = a(X_t, t)dt + b(X_t, t)dB_t.$$

where formally $dX_t = X_{t+dt} - X_t$. Roughly speaking, we express the evolution of the process in terms of differentials instead of derivatives since Brownian motion with probability one is non-differentiable. We remark that non-differentiability is intuitively suggested by Brownian Motion having a standard deviation over the time Δt of $\sqrt{Var[B_{\Delta t}]} = \sqrt{\Delta t}$. While differential notation circumvents the issue of having to differentiate Brownian motion, however, a a problem still arises in how we should "integrate" such equations. A natural approach is to try to construct the process by using the approximate expressions:

$$\Delta\tilde{X}_t = a(\tilde{X}_t, t)\Delta t + b(\tilde{X}_t, t)\Delta B_t,$$

where $\Delta\tilde{X}_t = \tilde{X}_{t+\Delta t} - \tilde{X}_t$ and $\Delta B_t = B_{t+\Delta t} - B_t$. Then by discretizing time $t_j = j\Delta t$ it seems natural to use a Riemann sum approximation by summing the approximate differential expression at times t_j to obtain:

$$\tilde{X}_T - \tilde{X}_0 = \sum_{j=1}^{M} a(\tilde{X}_{t_j^*}, t_j^*)\Delta t_j + \sum_{j=1}^{M} b(\tilde{X}_{t_j^*}, t_j^*)\Delta B_{t_j}.$$

However, it is important to note that the points at which we sample t_j^* can be chosen in a few different ways in such a construction. Since the scaling of Brownian motion over an increment is $\sqrt{\Delta t}$, choosing different sample values of t_j^* in the last summation, which involves b, can lead to a difference in the contribution of this summand term of order Δt, which when summed does not go to zero in the limit $\Delta t \to 0$. Formally this can be seen by Taylor expanding b for one of the sample values to express the function approximately at the other sample value and using that $\Delta B_t \sim \sqrt{\Delta t}$. In fact it can be shown more rigorously that indeed different choices of the sampling value t_j^* lead to different limiting values of the sums as $\Delta t \to 0$, see [2]). Therefore, some care must be taken in how the stochastic differential expressions are interpreted.

For our purposes, we shall consider Ito's interpretation of the differential equations, and work only with Ito's Integral, see [1;2]. Formally, the Ito Integral can be thought of as arising from a Riemann sum approximation where the sampling is done at the beginning of each interval, $t_j^* = t_j$:

$$\int_0^T f(B_s, s) dB_s = \lim_{\Delta t \to 0} \sum_{j=1}^{M} f(B_{t_j}, t_j) \Delta B_{t_j}.$$

For a more sophisticated and rigorous discussion of these important issues and the conditions required to ensure the Ito Integral exists, see [1;2].

As another consequence of the $B_{\Delta t} \sim \sqrt{\Delta t}$ scaling of Brownian motion, care must be taken when attempting to find differential expressions for functions of a stochastic processes $f(X_t)$, which naturally arise when attempting to recast a problem through some type of change of variables. For X_t given by the stochastic differential equation above, the stochastic process $Y_t = f(X_t, t)$, where f is sufficiently smooth, is given by Ito's Formula, which states that Y_t satisfies the stochastic differential equation:

$$df(X_t, t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dX_t \cdot dX_t$$

where the formal rule $dB_t \cdot dt = 0$ and $dB_t \cdot dB_t = dt$ are applied when expanding dX_t. In other words, Ito's Formula gives the change of variable:

$$dY_t = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} a(X_t, t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} b^2(X_t, t) \right) dt + \frac{\partial f}{\partial x} b(X_t, t) dB_t.$$

A good heuristic to remember the Ito Formula is that we expand the function f to second order in x to account for the contributions of the terms with $\sqrt{\Delta t}$ scaling of the Brownian Motion.

Another useful result in the calculus of stochastic processes is the Ito Isometry which states that:

$$\mathrm{E}\left(\int_0^T \int_0^T f(X_r, r) g(X_s, s) dB_s dB_r \right) \quad = \quad \mathrm{E}\left(\int_0^T f(X_s, s) g(X_s, s) ds \right).$$

This is useful in computing statistics of Ito Integrals in a number of circumstances. For example, in the case that $f = f(t)$ it can be shown that the Ito Integral:

$$\Gamma = \int_0^T f(s) dB_s$$

is a Gaussian random variable. In this case, the mean is always 0 while the variance σ^2 can be computed from Ito's Isometry by setting $g = f = f(t)$ to obtain:

$$\sigma^2 = \mathrm{E}\left(\left[\int_0^T f(s) dB_s \right]^2 \right) = \int_0^T f^2(s) ds.$$

The theory of stochastic processes also has a very rich connection with partial differential equations. Two results we summarize here are the following. Let X_s be the stochastic process defined above and

$$u(x, t; s) = \mathrm{E}^{(x,t)} \left(f(X_s, s) \right)$$

where in the notation $\mathrm{E}^{(x,t)}$ means that we start the process at $X_t = x$ at time t and let it evolve until time $s > t$. The expectation u then satisfies what is called the Backward-Kolomogorov PDE:

$$\frac{\partial u(x,t)}{\partial t} = -a(x,t) \frac{\partial u(x,t)}{\partial x} - \frac{1}{2} b^2(x,t) \frac{\partial^2 u(x,t)}{\partial x^2}.$$

Now, let $p(x,t;y,s)$ denote the probability density that the process X_t is at location x at time t given that we started the process at location y at time s. The probability density p then satisfies what is called the Forward-Kolomogorov PDE:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\cdot\left(a(x,t)p(x,t) - \frac{1}{2}b^2(x,t)\frac{\partial p(x,t)}{\partial x}\right).$$

Example: (Brownian Motion)
For Brownian motion $X_t = B_t$ we have $a = 0$, $b = 1$. In this case the Forward-Kolomogorov Equation for $p(x,t;y,s)$ becomes:

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}.$$

This is the heat equation, which has the well-known solution

$$p(x,t;y,s) = \frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{(x-y)^2}{2(t-s)}\right).$$

Derivation of Black-Scholes-Merton PDE I

Using the results of stochastic calculus above and appealing to the notion of no arbitrage we can derive a PDE to describe the Black-Scholes-Merton value of an option. We shall proceed by considering a portfolio Π which consists of an option and a position in the underlying asset which hedges the uncertainty of future price movements of the underlying asset. More specifically let us consider the portfolio:

$$\Pi = V - \Delta\cdot S_t$$

where Δ is to be determined. Since the stock price has lognormal dynamics satisfying the SDE:

$$dS_t = rS_t dt + \sigma S_t dB_t$$

we have by Ito's Lemma that the change in the portfolio is given by:

$$\begin{aligned} d\Pi &= dV - \Delta\cdot dS_t \\ &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS_t + \frac{\partial^2 V}{\partial S^2}(dS_t)^2 - \Delta\cdot dS_t \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS_t. \end{aligned}$$

Now let $\Delta = \frac{\partial V}{\partial S}$ then the change in value of portfolio is no longer sensitive to changes in the underlying asset:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt.$$

This gives that the return of the portfolio to leading order in time is certain. More specifically, there is no risk in the return depending on price changes of the underlying asset S_t by our choice of Δ. By the principle of no arbitrage the return of the portfolio must be the same as the risk-free interest rate r. If the return where not r, say the return of Π were larger then one would simply take a loan at the risk-free rate r and buy the portfolio Π to obtain a guaranteed profit. Since this is ruled out by the principle of no arbitrage we must have:

$$\begin{aligned} d\Pi &= r\Pi dt \\ &= r(V - \Delta\cdot S_t)dt. \end{aligned}$$

Now the coefficients on dt must agree with the expression above obtained from Ito's Lemma. Equating the coefficients gives the following PDE for the option value V:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.$$

The PDE holds in the domain $t < T$ with the boundary condition $V(T, S) = f(S)$ given at maturity time T where the payoff $f(S)$ of the option is known. This equation is referred to as the Black-Scholes-Merton PDE. It is the analogue of the backward-induction procedure in the binomial tree in the continuum limit for time.

Derivation of the Black-Scholes-Merton PDE II

An alternative derivation of the Black-Scholes-Merton PDE can be obtained by considering the risk-neutral valuation formula for an option at time t and spot price s:

$$V(s,t) = e^{-r(T-t)} W(s,t)$$

with

$$W(s,t) = \mathrm{E}^{(s,t)}\left(Q(S_T)\right)$$

where $Q(s)$ is the payoff function at maturity time T. The stochastic process describing the stock dynamics is taken to be lognormal, which satisfies the SDE:

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

By the Backward-Kolomogorov PDE we have that $W(s,t)$ satisfies:

$$\frac{\partial W}{\partial t} = -rs\frac{\partial W}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 W}{\partial s^2}.$$

Now by differentiating $V(s,t)$ we have

$$\frac{\partial V}{\partial t}(s,t) \quad = \quad re^{-r(T-t)} W(s,t) + e^{-r(T-t)} \frac{\partial W}{\partial t}(s,t)$$

Now by substituting the expressions for $\frac{\partial W}{\partial t}(s,t)$ and using that multiplication by the exponential and differentiation in s commute we obtain the Black-Scholes-Merton PDE for the value of a European option at time t for the spot price s:

$$\frac{\partial V}{\partial t} \quad = \quad rV - rs\frac{\partial V}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2}.$$

This PDE has as its domain times $t < T$ with the boundary condition at time T given by $V(s,T) = Q(s)$.

A Numerical Method for the Black-Scholes-Merton PDE

There are many approaches which can be taken to numerically solve the Black-Scholes-Merton PDE. One approach is to transform the PDE to the standard heat equation. The heat equation can then be solved and transformed back to a solution of the Black-Scholes-Merton PDE. We shall show how this is done in stages. Let us first consider the transformation $(s,t) \to (x,\tau)$ given by:

$$\begin{aligned} s &= Ke^x \\ t &= T - \frac{2\tau}{\sigma^2}. \end{aligned}$$

By letting $v(x,\tau) = KV(s,t)$ we have:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (c-1)\frac{\partial v}{\partial x} - cv$$

where $c = \frac{2r}{\sigma^2}$. Now if we let $v(x,\tau) = e^{-\frac{1}{2}(c-1)x - \frac{1}{4}(c+1)^2\tau}u(x,\tau)$ then:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

The boundary condition for the original PDE, which is the payoff at maturity $V(s,T) = Q(s)$, becomes for the transformed system the boundary condition $u(x,0) = \frac{1}{K}e^{\frac{1}{2}(c-1)x + \frac{1}{4}(c+1)^2\tau}Q(Ke^x)$. An exact solution to the basic heat equation can be written in terms of the initial conditions as:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4t}}u(y,0)dy.$$

It is also desirable to develop numerical methods to construct approximate solutions of the PDE, which are then more readily amenable to handle extensions in pricing more sophisticated types of options. A numerical scheme commonly used is the implicit Crank-Nicholson Method, which is unconditionally stable. Space and time are discretized into a finite collection of lattice sites with the distance between lattice sites in space Δx and the distance between lattice sites in time $\Delta\tau$. We then approximate the function at each lattice site by $u(m\Delta x, n\Delta\tau) \approx u_m^n$, where u_m^n is the approximate value of the function obtained from the numerical computation. For the heat equation the Crank-Nicholson Method solves numerically using the scheme:

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = \frac{1}{2}\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{1}{2}\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}.$$

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